Supplementary Material for Integrated Low Rank Based Discriminative Feature Learning for Recognition

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I. PROOF OF THEOREM 3.2

Before we prove Theorem 3.2, we introduce a lemma, which is described as follows.

Lemma 1.1 ([1], [2]): Suppose that \( f: \mathbb{R}^m \rightarrow \mathbb{R} \) is a continuously differentiable function with Lipschitz continuous gradient whose Lipschitz constant is \( L \). Then for any \( x, y \in \mathbb{R}^m \) and \( \gamma \geq L \),

\[
    f(x) \leq f(y) + \langle x - y, \nabla f(y) \rangle + \frac{\gamma}{2} \| x - y \|_2^2.
\]

(1)

Now, we prove Theorem 3.2.

Proof Let \( f(Q_i) = \| H_i - Q_i \|_2^2 \) (\( i = 1, \cdots, r \)). It is easy to check that \( f(Q_i) \) satisfies Lemma 1.1 \( \nabla f(Q_i) = -2(H_i - Q_i) \) and \( L = 2 \). We first prove the following inequality:

\[
\langle Q_{i+1}^k - Q_i^k, \nabla f(Q_i^k) \rangle + \frac{\gamma}{2} \| Q_{i+1}^k - Q_i^k \|_2^2 + \frac{\alpha}{\beta_i^k} \| Q_{i+1}^k \|_2^2 + \frac{\alpha}{\beta_i^k} \| Q_i^k \|_2^2 \leq \frac{\alpha}{\beta_i^k} \| Q_i^k \|_2^2, \quad (\forall k, i),
\]

(2)

where \( k \) denotes the number of iterations. Define \( \beta_k = \frac{(g_i^{-1})^2}{(g_i^{-1})^2 + \alpha} \), then \( Q_i^k = \frac{(g_i^{-1})^2}{(g_i^{-1})^2 + \alpha} H_i = \beta_k H_i \). We consider inequality (2).

\[
c = \langle Q_{i+1}^k - Q_i^k, \nabla f(Q_i^k) \rangle + \frac{\gamma}{2} \| Q_{i+1}^k - Q_i^k \|_2^2 + \frac{\alpha}{\beta_i^k} \| Q_{i+1}^k \|_2^2 - \frac{\alpha}{\beta_i^k} \| Q_i^k \|_2^2
\]

\[
= \Bigg( Q_{i+1}^k - Q_i^k, \nabla f(Q_i^k) \Bigg) + \frac{\gamma}{2} \| Q_{i+1}^k - Q_i^k \|_2^2 + \frac{\alpha}{\beta_i^k} \| Q_{i+1}^k \|_2^2 + \frac{\alpha}{\beta_i^k} \| Q_i^k \|_2^2
\]

\[
= (\beta_{k+1} - \beta_k) \left[ 2 - \frac{\gamma}{2} + \frac{\alpha}{(g_i^k)^2} \right] \beta_k + \left( \frac{\gamma}{2} + \frac{\alpha}{(g_i^k)^2} \right) \left[ \beta_{k+1} - 2 \right] \| H_i \|_2^2
\]

(3)

So if \( \gamma \) satisfies \( L = 2 < \gamma \leq 4 \), we have that \( c \leq 0 \), i.e., inequality (2) holds.

Note that \( g_i^{k+1} \) is the optimal solution to problem (4):

\[
g_i^{k+1} = \arg \min \sum_{i=1}^r \frac{\alpha}{g_i^k} \| Q_i^{k+1} \|_2^2.
\]

(4)

So the following inequality holds.

\[
\sum_{i=1}^r \frac{\alpha}{(g_i^k)^2} \| Q_i^{k+1} \|_2^2 \geq \sum_{i=1}^r \frac{\alpha}{(g_i^k)^2} \| Q_i^{k+1} \|_2^2.
\]

(5)

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Then, combining inequalities (2) and (5), we can further obtain the following inequality:

\[
- \sum_{i=1}^{r} (Q_{i}^{k+1} - Q_{i}^{k}, \nabla f(Q_{i}^{k})) \geq \sum_{i=1}^{r} \left( \frac{\gamma}{2} \|Q_{i}^{k+1} - Q_{i}^{k}\|^2 \right) + \frac{\alpha}{(g_{1}^{k})^2} \|Q_{i}^{k+1}\|^2 - \frac{\alpha}{(g_{1}^{k})^2} \|Q_{i}^{k}\|^2 \right)
\]

\[
\geq \sum_{i=1}^{r} \left( \frac{\gamma}{2} \|Q_{i}^{k+1} - Q_{i}^{k}\|^2 \right) + \frac{\alpha}{(g_{1}^{k+1})^2} \|Q_{i}^{k+1}\|^2 - \frac{\alpha}{(g_{1}^{k+1})^2} \|Q_{i}^{k}\|^2 \right).
\]

(6)

Since \( f(Q_{i}) \) (i = 1, \cdots, r) satisfies Lemma 1.4 the following inequality holds:

\[
\sum_{i=1}^{r} (f(Q_{i}^{k}) - f(Q_{i}^{k+1})) \geq \sum_{i=1}^{r} \left((-Q_{i}^{k+1} - Q_{i}^{k}, \nabla f(Q_{i}^{k})) - \frac{L}{2} \|Q_{i}^{k+1} - Q_{i}^{k}\|^2 \right)
\]

\[
\geq \sum_{i=1}^{r} \left( \frac{\alpha}{(g_{1}^{k})^2} \|Q_{i}^{k+1}\|^2 - \frac{\alpha}{(g_{1}^{k+1})^2} \|Q_{i}^{k}\|^2 \right).
\]

(7)

Thus, we can obtain the following inequality:

\[
F(Q^{k}, g^{k}) - F(Q^{k+1}, g^{k+1}) = \sum_{i=1}^{r} \left( f(Q_{i}^{k}) - f(Q_{i}^{k+1}) + \frac{\alpha}{(g_{1}^{k})^2} \|Q_{i}^{k+1}\|^2 - \frac{\alpha}{(g_{1}^{k+1})^2} \|Q_{i}^{k}\|^2 \right)
\]

\[
\geq \frac{\gamma - L}{2} \sum_{i=1}^{r} \|Q_{i}^{k+1} - Q_{i}^{k}\|^2 = \frac{\gamma - L}{2} \|Q^{k+1} - Q^{k}\|^2 \geq 0.
\]

(8)

Therefore, \( F(Q^{k}, g^{k}) \) is monotonically decreasing. So \( F(Q^{k}, g^{k}) = \sum_{i=1}^{r} \left( \|Q_{i}^{k+1} - Q_{i}^{k}\|^2 \right) \leq F(Q^{1}, g^{1}) \). Thus \( \{Q^{k}\} \) is bounded. Summing all the inequality (6) for all \( k \geq 1 \), we obtain

\[
F(Q^{1}, g^{1}) - F(Q^{k+1}, g^{k+1}) \geq \frac{\gamma - L}{2} \sum_{j=1}^{k} \|Q^{j+1} - Q^{j}\|^2 \geq 0.
\]

(9)

As \( \gamma > L \), the above implies that \( \lim_{k \to \infty} \|Q^{k+1} - Q^{k}\|^2 = 0 \). As \( g^{k} = \frac{\|Q^{k+1}\|}{\sum_{i=1}^{r} \|Q_{i}^{k}\|^2} \) (\( \forall i = 1, \cdots, r \)), \( \lim_{k \to \infty} \|g^{k+1} - g^{k}\|^2 = 0 \). As \( \sum_{i=1}^{r} g_{i} = t > 0 \) and \( g_{i} \geq 0 \), the sequence \( \{g_{k}\} \) is bounded.

II. PROOF OF THEOREM 3.3

Proof In Theorem 3.2, we have proved that the sequence \( \{Q^{k}, g^{k}\} \) is bounded. For any accumulation point \((Q^{*}, g^{*})\) of \( \{Q^{k}, g^{k}\} \), suppose a subsequence \( (Q^{k_{j}}, g^{k_{j}}) \) fulfills \( \lim_{j \to \infty} Q^{k_{j}} = Q^{*} \) and \( \lim_{j \to \infty} g^{k_{j}} = g^{*} \). In each iteration, we denote

\[
\tau^{k} = \frac{2\alpha}{(g_{1}^{k})^2} \left( \sum_{i=1}^{r} \|Q_{i}^{k}\|^2 \right)^{3}.
\]

(10)

Then there exists \( \tau^{*} \) such that \( \lim_{j \to \infty} \tau^{k_{j}} = \tau^{*} \). In our iteration process, \( \nabla_{Q} F(Q^{k_{j}+1}, g^{k_{j}}) = 0 \), \( \nabla_{g} F(Q^{k_{j}+1}, g^{k_{j}+1}) + \tau^{k_{j}+1} = 0 \), and \( \sum_{i=1}^{r} g_{i} = t \). Letting \( j \to \infty \), we have \( \nabla_{Q} F(Q^{*}, g^{*}) = 0 \), \( \nabla_{g} F(Q^{*}, g^{*}) + \tau^{*} = 0 \), and \( \sum_{i=1}^{r} g_{i} = t \). So \((Q^{*}, g^{*})\) is a KKT point.

III. FAST ALGORITHM FOR ROBUST PCA

In this section, we introduce the \( \ell_{1} \)-filtering for solving Robust PCA problem. We sketch it below.

\( \ell_{1} \)-filtering first randomly samples a \( s_{r} \times s_{c} \) submatrix \( X^{s} \) from \( X \), where \( s_{r} > 1 \) and \( s_{c} > 1 \) are the row and column oversampling rates, respectively. For simplicity, we assume that \( X^{s} \) is at the top left corner of matrix \( X \). Then accordingly, \( X, A, \) and \( E \) is partitioned into:

\[
X = \begin{bmatrix} X^{s} & X^{c} \\ X^{r} & X^{s} \end{bmatrix}, \quad A = \begin{bmatrix} A^{s} & A^{c} \\ A^{r} & A^{s} \end{bmatrix}, \quad E = \begin{bmatrix} E^{s} & E^{c} \\ E^{r} & E^{s} \end{bmatrix},
\]

(11)

where \( A \) is the low rank matrix we need to recover and \( E \) is the sparse error matrix.

Then \( \ell_{1} \)-filtering recovers \( A^{s} \), called the seed matrix, from \( X^{s} \) by solving a small-sized Robust PCA problem. Since \( s_{r} \times r \) and \( s_{c} \times c \) are both small as compared with \( m \) and \( n \), the computation of recovering \( A^{s} \) is much cheaper than recovering the whole \( A \).
Next, as the rank of $A$ and $A^*$ are both $r$, there must exist matrix $Q$ and $P$ satisfying the following equations:

$$A^c = A^* Q, \quad A^r = P^T A^*.$$  

(12)

Since the matrix $E$ is sparse, the matrices $E^c$ and $E^r$ are also sparse. So we can find matrix $Q$ and $P$ by minimizing the following problems:

$$\min_{E^c, Q} \| E^c \|_1, \quad \text{s.t.} \quad X^c = A^* Q + E^c,$$

and

$$\min_{E^r, P} \| E^r \|_1, \quad \text{s.t.} \quad X^r = P^T A^* + E^r,$$

(13)
(14)

respectively. For these two problems, using the alternating direction method (ADM) [4] to solve them is efficient. So we can get $P$ and $Q$, thus $A^c$ and $A^r$ can be also obtained. Finally, only $\hat{A}^*$ needs to be computed. By the low-rankness of $A$, we can obtain

$$\hat{A}^* = P^T A^* Q.$$  

(15)

In summary, the matrix $A$ can be recovered with a complexity of $O(r^2(d + m))$ at each iteration [3], which is much lower than $O(rmd)$ when $d$ and $m$ are large.

REFERENCES