Supplementary Material for Tensor Factorization for Low-Rank Tensor Completion

Pan Zhou, Canyi Lu, *Student Member, IEEE*, Zhouchen Lin, *Senior Member, IEEE*, and Chao Zhang, *Member, IEEE*

I. Proof of Lemma 2

Proof. Assume that $\mathrm{rank}_m(\mathcal{F}) = \boldsymbol{k}^{\mathcal{F}}$, where $\boldsymbol{k}_i^{\mathcal{F}} = \mathrm{rank}(\bar{\boldsymbol{F}}^{(i)})$ $(i = 1, \cdots, n_3)$. Then, we have $\hat{k} = \max(\boldsymbol{k}_1^{\mathcal{F}}, \cdots, \boldsymbol{k}_{n_3}^{\mathcal{F}})$. Since the rank of the matrix $\bar{\boldsymbol{F}}^{(i)}$ is $\boldsymbol{k}_i^{\mathcal{F}}$, it can be factorized into the matrix product form $\bar{\boldsymbol{F}}^{(i)} = \hat{\boldsymbol{G}}^{(i)}\hat{\boldsymbol{H}}^{(i)}$, where $\hat{\boldsymbol{G}}^{(i)} \in \mathbb{C}^{n_1 \times \boldsymbol{k}_i^{\mathcal{F}}}$ and $\hat{\boldsymbol{H}}^{(i)} \in \mathbb{C}^{\boldsymbol{k}_i^{\mathcal{F}} \times n_2}$ are the i-th block diagonal matrices of $\hat{\boldsymbol{G}} \in \mathbb{C}^{n_1 n_3 \times (\sum_{i=1}^{n_3} \boldsymbol{k}_i^{\mathcal{F}})}$ and $\hat{\boldsymbol{H}} \in \mathbb{C}^{(\sum_{i=1}^{n_3} \boldsymbol{k}_i^{\mathcal{F}}) \times n_2 n_3}$, respectively, and they meet $\mathrm{rank}(\hat{\boldsymbol{G}}^{(i)}) = \mathrm{rank}(\hat{\boldsymbol{H}}^{(i)}) = \boldsymbol{k}_i^{\mathcal{F}}$. Then, let $\bar{\boldsymbol{G}}^{(i)} = [\hat{\boldsymbol{G}}^{(i)}, \mathbf{0}] \in \mathbb{C}^{n_1 \times \hat{k}}$ and $\bar{\boldsymbol{H}}^{(i)} = [\hat{\boldsymbol{H}}^{(i)}; \mathbf{0}] \in \mathbb{C}^{\hat{k} \times n_2}$, where $\bar{\boldsymbol{G}}^{(i)} \in \mathbb{C}^{n_1 \times \hat{k}}$ and $\hat{\boldsymbol{H}}^{(i)} \in \mathbb{C}^{\hat{k} \times n_2}$ are the i-th block diagonal matrices of $\bar{\boldsymbol{G}} \in \mathbb{C}^{n_1 n_3 \times \hat{k} n_3}$ and $\bar{\boldsymbol{H}} \in \mathbb{C}^{\hat{k} n_3 \times n_2 n_3}$, respectively. Therefore, we have $\bar{\boldsymbol{C}} = \hat{\boldsymbol{G}}\hat{\boldsymbol{H}} = \bar{\boldsymbol{G}}\bar{\boldsymbol{H}}$. From Lemma 1, we know that for any three tensors of proper sizes, $\bar{\boldsymbol{E}} = \bar{\boldsymbol{X}}\bar{\boldsymbol{Y}}$ and $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{X}} * \boldsymbol{\mathcal{Y}}$ are equivalent. Therefore, we can obtain $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{G}} * \boldsymbol{\mathcal{H}}$, where $\boldsymbol{\mathcal{G}} \in \mathbb{R}^{n_1 \times \hat{k} \times n_3}$ and $\boldsymbol{\mathcal{H}} \in \mathbb{R}^{\hat{k} \times n_2 \times n_3}$ are two tensors of smaller sizes and they meet $\mathrm{rank}_{\mathrm{t}}(\boldsymbol{\mathcal{G}}) = \mathrm{rank}_{\mathrm{t}}(\boldsymbol{\mathcal{H}}) = \hat{k}$.

Now we prove the second property. Assume that $\mathrm{rank}_m(\boldsymbol{\mathcal{A}}) = \boldsymbol{r}^{\boldsymbol{\mathcal{A}}}$ and $\mathrm{rank}_t(\boldsymbol{\mathcal{A}}) = \hat{r}^{\boldsymbol{\mathcal{A}}}$, where $\boldsymbol{r}_i^{\boldsymbol{\mathcal{A}}} = \mathrm{rank}(\bar{\boldsymbol{A}}^{(i)})$ $(i = 1, \cdots, n_3)$ and $\hat{r}^{\boldsymbol{\mathcal{A}}} = \max(\boldsymbol{r}_1^{\boldsymbol{\mathcal{A}}}, \cdots, \boldsymbol{r}_{n_3}^{\boldsymbol{\mathcal{A}}})$. Let $\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{A}} * \boldsymbol{\mathcal{B}}$. Similarly, suppose that $\mathrm{rank}_m(\boldsymbol{\mathcal{B}}) = \boldsymbol{r}^{\boldsymbol{\mathcal{B}}}$, $\mathrm{rank}_t(\boldsymbol{\mathcal{B}}) = \hat{r}^{\boldsymbol{\mathcal{B}}}$, $\mathrm{rank}_m(\boldsymbol{\mathcal{Z}}) = \boldsymbol{r}^{\boldsymbol{\mathcal{Z}}}$, and $\mathrm{rank}_t(\boldsymbol{\mathcal{Z}}) = \hat{r}^{\boldsymbol{\mathcal{Z}}}$. On the other hand, if $\boldsymbol{M} \in \mathbb{C}^{n_5 \times n_6}$ and $\boldsymbol{N} \in \mathbb{C}^{n_6 \times n_7}$ are two matrices, then we have $\mathrm{rank}(\boldsymbol{M}\boldsymbol{N}) \le \min(\mathrm{rank}(\boldsymbol{M}), \mathrm{rank}(\boldsymbol{N}))$. Thus, we have $\boldsymbol{r}_i^{\boldsymbol{\mathcal{Z}}} = \mathrm{rank}(\bar{\boldsymbol{Z}}^{(i)}) = \mathrm{rank}(\bar{\boldsymbol{A}}^{(i)}\bar{\boldsymbol{B}}^{(i)}) \le \min(\mathrm{rank}(\bar{\boldsymbol{A}}^{(i)}), \mathrm{rank}(\bar{\boldsymbol{B}}^{(i)})) = \min(\boldsymbol{r}_i^{\boldsymbol{\mathcal{A}}}, \boldsymbol{r}_i^{\boldsymbol{\mathcal{B}}})$. We can further obtain that $\hat{r}^{\boldsymbol{\mathcal{Z}}} = \max(r_1^{\boldsymbol{\mathcal{Z}}}, \cdots, r_{n_3}^{\boldsymbol{\mathcal{Z}}}) \le \min(\hat{r}^{\boldsymbol{\mathcal{A}}}, \hat{r}^{\boldsymbol{\mathcal{B}}})$. So the inequality $\mathrm{rank}_t(\boldsymbol{\mathcal{A}} * \boldsymbol{\mathcal{B}}) \le \min(\mathrm{rank}_t(\boldsymbol{\mathcal{A}}), \mathrm{rank}_t(\boldsymbol{\mathcal{B}}))$ in Lemma 2 holds. ☐

II. Proof of Theorem 2

Before we prove Theorem 2, we first present two lemmas. Since $\hat{\boldsymbol{X}}^{(i)}$ and $\hat{\boldsymbol{Y}}^{(i)}$ are the i-th block diagonal matrices of $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{Y}}$, respectively, for brevity, we rewrite the Eq. (6) and (7) as $\hat{\boldsymbol{X}}^{k+1} = \bar{\boldsymbol{C}}^k(\hat{\boldsymbol{Y}}^k)^* \left(\hat{\boldsymbol{Y}}^k(\hat{\boldsymbol{Y}}^k)^*\right)^\dagger$ and $\hat{\boldsymbol{Y}}^{k+1} = \left((\hat{\boldsymbol{X}}^{k+1})^*\hat{\boldsymbol{X}}^{k+1}\right)^\dagger (\hat{\boldsymbol{X}}^{k+1})^*\bar{\boldsymbol{C}}^k$, respectively.

Lemma 3. *Assume that the sequence $\{(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k, \boldsymbol{\mathcal{C}}^k)\}$ is generated by Algorithm 1, i.e., they meet $\hat{\boldsymbol{X}}^{k+1} = \bar{\boldsymbol{C}}^k(\hat{\boldsymbol{Y}}^k)^* \left(\hat{\boldsymbol{Y}}^k(\hat{\boldsymbol{Y}}^k)^*\right)^\dagger \in \mathbb{C}^{n_1 n_3 \times \sum_{i=1}^{n_3} \boldsymbol{r}_i^k}$ and $\hat{\boldsymbol{Y}}^{k+1} = \left((\hat{\boldsymbol{X}}^{k+1})^*\hat{\boldsymbol{X}}^{k+1}\right)^\dagger (\hat{\boldsymbol{X}}^{k+1})^*\bar{\boldsymbol{C}}^k \in \mathbb{C}^{\sum_{i=1}^{n_3} \boldsymbol{r}_i^k \times n_2 n_3}$. Suppose that $\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{\Sigma}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{V}_{\hat{\boldsymbol{X}}^{k+1}}^*$ and $\boldsymbol{U}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{\Sigma}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}^*$ are the skinny SVD of $\hat{\boldsymbol{X}}^{k+1}$ and $\hat{\boldsymbol{Y}}^k$, respectively. Then the sequence $\{(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k, \boldsymbol{\mathcal{C}}^k)\}$ satisfies the following equations:*

$$\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k\|_F^2 = \|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}^*(\bar{\boldsymbol{C}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k)\|_F^2 + \|(\boldsymbol{I}_{n_1 n_3} - \boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}^*)(\bar{\boldsymbol{C}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}^*\|_F^2 \tag{22}$$

and

$$\|\hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k - \bar{\boldsymbol{C}}^k\|_F^2 - \|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^k\|_F^2 = \|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k\|_F^2. \tag{23}$$

Proof. Since $\hat{\boldsymbol{X}}^{k+1} = \boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}^*\hat{\boldsymbol{X}}^{k+1}$ and $\hat{\boldsymbol{Y}}^k = \hat{\boldsymbol{Y}}^k\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}^*$, we have

$$\begin{aligned} \hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k &= \bar{\boldsymbol{C}}^k(\hat{\boldsymbol{Y}}^k)^* \left(\hat{\boldsymbol{Y}}^k(\hat{\boldsymbol{Y}}^k)^*\right)^\dagger \hat{\boldsymbol{Y}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k \\ &= (\bar{\boldsymbol{C}}^k - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}\boldsymbol{V}_{\hat{\boldsymbol{Y}}^k}^*. \end{aligned} \tag{24}$$

P. Zhou, Z. Lin, and C. Zhang are with Key Lab. of Machine Perception (MoE), School of EECS, Peking University, P. R. China. Z. Lin and C. Zhang are also with Cooperative Medianet Innovation Center, Shanghai, China. P. Zhou is now with Department of Electrical & Computer Engineering, National University of Singapore, Singapore. (e-mails: pzhou@pku.edu.cn, zlin@pku.edu.cn, and chzhang@cis.pku.edu.cn).

C. Lu is with the Department of Electrical and Computer Engineering, National University of Singapore, Singapore (e-mail: canyilu@gmail.com).

On the other hand, we can obtain the following equation:

$$\begin{aligned}\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}=&\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}\\=&\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}\hat{\boldsymbol{X}}^{k+1}\left((\hat{\boldsymbol{X}}^{k+1})^{*}\hat{\boldsymbol{X}}^{k+1}\right)^{\dagger}(\hat{\boldsymbol{X}}^{k+1})^{*}\bar{\boldsymbol{C}}^{k}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}(\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}+\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\\=&\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}\bar{\boldsymbol{C}}^{k}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}\left(\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}+(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^{k}}\boldsymbol{V}^{*}_{\hat{\boldsymbol{Y}}^{k}}\right)\\=&\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})(\boldsymbol{I}_{n_2n_3}-\boldsymbol{V}_{\hat{\boldsymbol{Y}}^{k}}\boldsymbol{V}^{*}_{\hat{\boldsymbol{Y}}^{k}}).\end{aligned}\tag{25}$$

Then the following equation holds:

$$\begin{aligned}\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}=&\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}+\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\\=&(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^{k}}\boldsymbol{V}^{*}_{\hat{\boldsymbol{Y}}^{k}}+\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}).\end{aligned}\tag{26}$$

Note that $\left\langle(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^{k}}\boldsymbol{V}^{*}_{\hat{\boldsymbol{Y}}^{k}},\ \boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\right\rangle=0$, since they are orthogonal to each other. Thus, we can obtain

$$\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2=\|(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^{k}}\boldsymbol{V}^{*}_{\hat{\boldsymbol{Y}}^{k}}\|_F^2+\|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\|_F^2.\tag{27}$$

Therefore, Eq. (22) holds. We can further establish the following equation:

$$\begin{aligned}&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\bar{\boldsymbol{C}}^{k}\|_F^2\\=&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}+\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^{k}\|_F^2\\=&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2+\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^{k}\|_F^2+2\left\langle\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^{k},\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\right\rangle\\=&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2+\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^{k}\|_F^2+2\left\langle\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^{k},(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^{k}}\boldsymbol{V}^{*}_{\hat{\boldsymbol{Y}}^{k}}\right.\\&\left.+\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\right\rangle\\=&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2+\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^{k}\|_F^2-2\left(\|(\boldsymbol{I}_{n_1n_3}-\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}})(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\boldsymbol{V}_{\hat{\boldsymbol{Y}}^{k}}\boldsymbol{V}^{*}_{\hat{\boldsymbol{Y}}^{k}}\|_F^2\right.\\&\left.+\|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{k+1}}\boldsymbol{U}^{*}_{\hat{\boldsymbol{X}}^{k+1}}(\bar{\boldsymbol{C}}^{k}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k})\|_F^2\right)\\=&\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2+\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^{k}\|_F^2-2\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2\\=&\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^{k}\|_F^2-\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2.\end{aligned}\tag{28}$$

Therefore, Eq. (23) holds. □

Then, we present anther lemma, which will be used later.

Lemma 4. *Suppose that* $\boldsymbol{\mathcal{A}}\in\mathbb{R}^{n_1\times n_2\times n_3}$, $\boldsymbol{\mathcal{B}}\in\mathbb{R}^{n_2\times n_4\times n_3}$, $\boldsymbol{\mathcal{F}}\in\mathbb{R}^{n_1\times n_4\times n_3}$ *and* $\boldsymbol{\mathcal{H}}\in\mathbb{R}^{n_1\times n_4\times n_3}$ *are four tensors. If they satisfy the following inequality:*

$$\|\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{F}}\|_F^2\le\|\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{H}}\|_F^2,\tag{29}$$

then we have

$$\|\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{F}}\|_F^2\le\|\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{H}}\|_F^2.\tag{30}$$

Proof. From Lemma 1 in the paper, we know that $\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{F}}$ and $\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{F}}$ are equivalent to each other. $\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{H}}$ and $\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{H}}$ are also equivalent. Thus, we can obtain $\|\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{F}}\|_F^2=\frac{1}{n_3}\|\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{F}}\|_F^2$ and $\|\boldsymbol{\mathcal{A}}*\boldsymbol{\mathcal{B}}-\boldsymbol{\mathcal{H}}\|_F^2=\frac{1}{n_3}\|\bar{\boldsymbol{A}}\bar{\boldsymbol{B}}-\bar{\boldsymbol{H}}\|_F^2$. Thus, if inequality (29) holds, then inequality (30) holds. □

Now, we prove Theorem 2.

Proof. Assume that $f(\hat{\boldsymbol{X}},\hat{\boldsymbol{Y}},\boldsymbol{\mathcal{C}})=\frac{1}{2n_3}\|\hat{\boldsymbol{X}}\hat{\boldsymbol{Y}}-\bar{\boldsymbol{C}}\|$ is the objective function. From Lemma 3, the following equation holds.

$$\begin{aligned}f(\hat{\boldsymbol{X}}^{k},\hat{\boldsymbol{Y}}^{k},\boldsymbol{\mathcal{C}}^{k})-f(\hat{\boldsymbol{X}}^{k+1},\hat{\boldsymbol{Y}}^{k+1},\boldsymbol{\mathcal{C}}^{k})=&\frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}-\bar{\boldsymbol{C}}^{k}\|_F^2-\frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\bar{\boldsymbol{C}}^{k}\|_F^2\\=&\frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1}-\hat{\boldsymbol{X}}^{k}\hat{\boldsymbol{Y}}^{k}\|_F^2.\end{aligned}\tag{31}$$

On the other hand, we note that $\mathcal{C}^{k+1}$ is the optimal solution to problem (5) in the paper:

$$\mathcal{C}^{k+1} = \underset{P_\Omega(\mathcal{C}-\mathcal{M})=\mathbf{0}}{\operatorname{argmin}} \|\mathcal{X}^{k+1} * \mathcal{Y}^{k+1} - \mathcal{C}\|_F^2. \tag{32}$$

At the same time, we note that $P_\Omega(\mathcal{C}^k - \mathcal{M}) = 0$, *i.e.*, $\mathcal{C}^k$ is a feasible solution to problem (32). So the following inequality holds.

$$\|\mathcal{X}^{k+1} * \mathcal{Y}^{k+1} - \mathcal{C}^{k+1}\|_F^2 \le \|\mathcal{X}^{k+1} * \mathcal{Y}^{k+1} - \mathcal{C}^k\|_F^2, \tag{33}$$

From Lemma 4, we can obtain

$$\|\bar{\boldsymbol{X}}^{k+1}\bar{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k+1}\|_F^2 \le \|\bar{\boldsymbol{X}}^{k+1}\bar{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^k\|_F^2. \tag{34}$$

Since $\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} = \bar{\boldsymbol{X}}^{k+1}\bar{\boldsymbol{Y}}^{k+1}$, we have

$$\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k+1}\|_F^2 \le \|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^k\|_F^2. \tag{35}$$

Then, it follows that

$$\begin{aligned} & f(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k, \mathcal{C}^k) - f(\hat{\boldsymbol{X}}^{k+1}, \hat{\boldsymbol{Y}}^{k+1}, \mathcal{C}^{k+1}) \\ =& \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k - \bar{\boldsymbol{C}}^k\|_F^2 - \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k+1}\|_F^2 \\ =& \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k - \bar{\boldsymbol{C}}^k\|_F^2 - \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^k\|_F^2 + \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^k\|_F^2 - \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \bar{\boldsymbol{C}}^{k+1}\|_F^2 \\ \ge& \frac{1}{2n_3}\|\hat{\boldsymbol{X}}^{k+1}\hat{\boldsymbol{Y}}^{k+1} - \hat{\boldsymbol{X}}^k\hat{\boldsymbol{Y}}^k\|_F^2. \end{aligned} \tag{36}$$

Summing all the inequality (36) for all k, we obtain

$$f(\hat{\boldsymbol{X}}^1, \hat{\boldsymbol{Y}}^1, \mathcal{C}^1) - f(\hat{\boldsymbol{X}}^n, \hat{\boldsymbol{Y}}^n, \mathcal{C}^n) = \frac{1}{2n_3}\sum_{i=1}^{n}\|\hat{\boldsymbol{X}}^{i+1}\hat{\boldsymbol{Y}}^{i+1} - \hat{\boldsymbol{X}}^i\hat{\boldsymbol{Y}}^i\|_F^2 < +\infty. \tag{37}$$

Thus, we can obtain the following equation:

$$\lim_{n\to+\infty}\|\hat{\boldsymbol{X}}^{n+1}\hat{\boldsymbol{Y}}^{n+1} - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n\|_F^2 = 0, \tag{38}$$

Assume that $\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{\Sigma}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{V}^*_{\hat{\boldsymbol{X}}^{n+1}}$ and $\boldsymbol{U}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{\Sigma}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}$ are the skinny SVD of $\hat{\boldsymbol{X}}^{n+1}$ and $\hat{\boldsymbol{Y}}^n$, respectively. From Lemma 3, we can further obtain

$$\lim_{n\to+\infty}\|(\boldsymbol{I}_{n_1n_3} - \boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}})(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}\|_F^2 + \|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\|_F^2 = 0, \tag{39}$$

So, the following two equations hold:

$$\lim_{n\to+\infty}\|(\boldsymbol{I}_{n_1n_3} - \boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}})(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}\|_F^2 = 0 \tag{40}$$

and

$$\lim_{n\to+\infty}\|\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\|_F^2 = 0. \tag{41}$$

We can further establish the following equations:

$$\lim_{n\to+\infty}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n) = \mathbf{0}. \tag{42}$$

Since $\hat{\boldsymbol{Y}}^n$ is bounded, $\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}$ is bounded. Thus, we can establish the following equation:

$$\lim_{n\to+\infty}\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n} = \mathbf{0}. \tag{43}$$

So we can obtain

$$\mathbf{0} = \lim_{n\to+\infty}(\boldsymbol{I}_{n_1n_3} - \boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}})(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n} = \lim_{n\to+\infty}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}. \tag{44}$$

Since $(\hat{\boldsymbol{Y}}^n)^* = \boldsymbol{V}_{\hat{\boldsymbol{Y}}_n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}_n}(\hat{\boldsymbol{Y}}_n)^*$, $(\hat{\boldsymbol{X}}^{n+1})^* = (\hat{\boldsymbol{X}}^{n+1})^*\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}$, and $\hat{\boldsymbol{Y}}^n$, $\hat{\boldsymbol{X}}^{n+1}$ are bounded, we have

$$\mathbf{0} = \lim_{n\to+\infty}(\hat{\boldsymbol{X}}^{n+1})^*\boldsymbol{U}_{\hat{\boldsymbol{X}}^{n+1}}\boldsymbol{U}^*_{\hat{\boldsymbol{X}}^{n+1}}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n) = \lim_{n\to+\infty}(\hat{\boldsymbol{X}}^{n+1})^*(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n) \tag{45}$$

and

$$\mathbf{0} = \lim_{n\to+\infty}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)\boldsymbol{V}_{\hat{\boldsymbol{Y}}^n}\boldsymbol{V}^*_{\hat{\boldsymbol{Y}}^n}(\hat{\boldsymbol{Y}}^n)^* = \lim_{n\to+\infty}(\bar{\boldsymbol{C}}^n - \hat{\boldsymbol{X}}^n\hat{\boldsymbol{Y}}^n)(\hat{\boldsymbol{Y}}^n)^*. \tag{46}$$

Since the sequence $\{(\hat{\boldsymbol{X}}^k, \hat{\boldsymbol{Y}}^k, \boldsymbol{\mathcal{C}}^k)\}$ generated by our algorithm is bounded, there is a subsequence $\{(\hat{\boldsymbol{X}}^{k_j}, \hat{\boldsymbol{Y}}^{k_j}, \boldsymbol{\mathcal{C}}^{k_j})\}$ that converges to a point $(\hat{\boldsymbol{X}}_\star, \hat{\boldsymbol{Y}}_\star, \boldsymbol{\mathcal{C}}_\star)$. Therefore, the following two equations hold:

$$(\bar{\boldsymbol{C}}_\star - \hat{\boldsymbol{X}}_\star\hat{\boldsymbol{Y}}_\star)(\hat{\boldsymbol{Y}}_\star)^* = \mathbf{0}, \tag{47}$$

$$(\hat{\boldsymbol{X}}_\star)_*(\bar{\boldsymbol{C}}_\star - \hat{\boldsymbol{X}}_\star\hat{\boldsymbol{Y}}_\star) = \mathbf{0}. \tag{48}$$

On the other hand, we update $\boldsymbol{\mathcal{C}}^{k+1} = \boldsymbol{\mathcal{X}}^k * \boldsymbol{\mathcal{Y}}^k + P_\Omega(\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{X}}^k * \boldsymbol{\mathcal{Y}}^k)$ at each iteration. Thus, $\boldsymbol{\mathcal{C}}_\star$ always satisfies the following two equations.

$$\begin{aligned} &P_{\Omega^c}(\boldsymbol{\mathcal{C}}_\star - \boldsymbol{\mathcal{X}}_\star * \boldsymbol{\mathcal{Y}}_\star) = \mathbf{0}, \\ &P_\Omega(\boldsymbol{\mathcal{C}}_\star - \boldsymbol{\mathcal{M}}) = \mathbf{0}. \end{aligned} \tag{49}$$

And we can always find $\boldsymbol{\mathcal{Q}}_\star$ that meets the following equations.

$$P_\Omega(\boldsymbol{\mathcal{C}}_\star - \boldsymbol{\mathcal{X}}_\star * \boldsymbol{\mathcal{Y}}_\star) + \boldsymbol{\mathcal{Q}}_\star = \mathbf{0}. \tag{50}$$

So $(\hat{\boldsymbol{X}}_\star, \hat{\boldsymbol{Y}}_\star, \boldsymbol{\mathcal{C}}_\star)$ is a KKT point of problem (13) in the paper. □